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Semi-Hyperbolic Mappings, Condensing Operators, and Neutral Delay Equations*

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Semi-hyperbolic mappings in Banach spaces are Lipschitz continuous and not necessarily invertible. Like hyperbolic mappings, they involve a splitting into stable and unstable spaces, but a slight leakage from the strict invariance of the spaces is

bolic if and only if it is ψ-contracting and has no spectral values on the unit circle. A bishadowing result, which combines both direct and indirect forms of shadowing, is extended to semi-hyperbolic mappings in Banach spaces with locally condensing continuous comparison mappings. The result is applied to linear neutral delay equations with nonsmooth perturbations. © 1997 Academic Press

1. INTRODUCTION

The appropriate state space for a semi-dynamical system generated by a delay differential equation or a parabolic partial differential equation is an infinite dimensional function space. To be applicable to such systems the concept of hyperbolicity has thus been extended to Banach spaces [14, 19, 20], with noninvertibility of the mapping being a fundamental and necessary change. A related concept of semi-hyperbolicity was introduced in finite dimensions [6–9] to allow for nonsmoothness as well as noninvertibility. Another major difference is that it does not require the invariance of set under consideration or of the splitting subspaces, with the possible leakage from invariance in the latter case being described by four parameters called a split. A Banach space version of semi-hyperbolicity, first given

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in [3], will be used here. Unlike other infinite dimensional generalizations of hyperbolicity, it assumes that the unstable splitting subspaces are finite dimensional. This is satisfied in many important applications and allows stronger dynamical results to be established.

The effect of a semi-hyperbolic mapping in infinite dimensional Banach spaces is dominated by a coupling of finite dimensional expanding and infinite dimensional contracting behaviour. Such mappings need not be compact, but it will be shown here that they are locally ψ-condensing, in fact locally ψ-contracting, with respect to the Hausdorff measure of non-compactness ψ [5, 23]. In the linear case a classification of semi-hyperbolicity in terms of global ψ-contraction and explicit spectral bounds is also possible. This will be established in Sections 3 and 4 following a statement of the definition of semi-hyperbolicity in Section 2. A ψ-bishadowing theorem, which includes both direct and indirect forms of shadowing with locally ψ-condensing comparison mappings, will then be formulated and proved in Section 5, generalizing the bishadowing theorem in [3] involving compact comparison mappings. Finally, as a simple nontrivial application, this ψ-bishadowing result will be illustrated in the context of a hyperbolic linear neutral delay equation, for which the shift operator is semi-hyperbolic and the shift operators of nonsmooth perturbations of the delay equation are locally ψ-condensing.

2. SEMI-HYPERBOLIC MAPPINGS

Recall from [3] that a four-tuple $\mathbf{s} = (\lambda_s, \lambda_u, \mu_s, \mu_u)$ of nonnegative real numbers is called a *split* if

$$\lambda_s < 1 < \lambda_u \tag{2.1}$$

and

$$(1-\lambda_s)(\lambda_u - 1) > \mu_s \mu_u. \tag{2.2}$$

DEFINITION 1 (Semi-hyperbolicity). Let $\mathbf{s} = (\lambda_s, \lambda_u, \mu_s, \mu_u)$ be a split and K a subset of a Banach space $(E, \|\cdot\|_E)$. A locally Lipschitz mapping $f: E \to E$ is said to be $\mathbf{s}$-*semi-hyperbolic on the set* K if there exist positive real numbers k, δ, and an equivalent norm $\|\cdot\|$ such that for each $x \in K$ there exists a splitting (decomposition)

$$E = E_x^s \oplus E_x^u \tag{2.3}$$

with corresponding projectors P_x^s and P_x^u satisfying the following properties:

SH0. The space E_x^u is finite dimensional for all x and $\dim(E_x^u) = \dim(E_y^u)$ if $x, y \in K$.

SH1. $\sup_{x \in K}\{\|P_x^s\|, \|P_x^u\|\} \leqslant k$.

SH2. The inequalities

$$\|P_y^s(f(x+u+v) - f(x+\tilde{u}+v))\| \leqslant \lambda_s \|u - \tilde{u}\|, \tag{2.4}$$

$$\|P_y^s(f(x+u+v) - f(x+u+\tilde{v}))\| \leqslant \mu_s \|v - \tilde{v}\|, \tag{2.5}$$

$$\|P_y^u(f(x+u+v) - f(x+\tilde{u}+v))\| \leqslant \mu_u \|u - \tilde{u}\|, \tag{2.6}$$

$$\|P_y^u(f(x+u+v) - f(x+u+\tilde{v}))\| \geqslant \lambda_u \|v - \tilde{v}\| \tag{2.7}$$

hold for all $x, y \in K$ with $\|f(x) - y\| \leqslant \delta$ and all $u, \tilde{u} \in E_x^s$, $v, \tilde{v} \in E_x^u$ such that $\|u\|$, $\|\tilde{u}\|$, $\|v\|$, $\|\tilde{v}\| \leqslant \delta$.

Note that continuity in x of the splitting subspaces E_x^s, E_x^u or of the projectors P_x^s, P_x^u is not assumed here. Nor is invariance of the set K or of the splitting subspaces required, as is the case in the definition of hyperbolicity.

We mention two degenerate examples of semi-hyperbolic mappings. If the subspaces $E_x^u = \{0\}$ and $E_x^s = E$ for all $x \in K$ the semi-hyperbolic mapping is just a mapping which is locally contracting on some neighbourhood of K, whereas if $E_x^s = \{0\}$ and $E_x^u = E$ for all $x \in K$ it is locally expanding, the class of such mappings here being larger than those that are usually considered.

3. CONDENSING OPERATORS AND CONTRACTIONS

The *Hausdorff measure of noncompactness* $\psi(M)$ of a nonempty bounded subset M of E is defined by

$$\psi(M) = \inf\{r > 0\colon M \text{ can be covered by finitely many balls of radius } r\}.$$

Note that $\psi(M) = 0$ if and only if M is relatively compact. Other properties of ψ include

(1) monotonicity: if $M_1 \subseteq M_2$, then $\psi(M_1) \leqslant \psi(M_2)$;

(2) subadditivity: $\psi(M_1 + M_2) \leqslant \psi(M_1) + \psi(M_2)$;

(3) homogenity: $\psi(\gamma M) = |\gamma|\, \psi(M)$, $\gamma \in \mathbb{R}$;

(4) $\psi(\mathrm{conv}(M)) = \psi(M)$, where $\mathrm{conv}(M)$ denotes the closed convex hull of M;

(5) For each $\varepsilon > 0$ the set M possesses a finite $(\psi(M) + \varepsilon)$-net.

The first four properties here are taken from Theorem 7.2 in [5] (see also [23], Theorem 1.2.3), while the last follows from the definition.

A continuous mapping $f\colon X \to E$, where $X \subseteq E$, is said to be *ψ-condensing* on X if

$$\psi(f(M)) < \psi(M)$$

whenever $M \subset X$ is bounded and not relatively compact. A condensing mapping f on X is called *χ-ψ-contracting* on X, where $0 \leqslant \chi < 1$, if

$$\psi(f(M)) \leqslant \chi\psi(M)$$

for all bounded subsets M of X.

A continuous mapping $f\colon E \to E$ is said to be *δ-locally ψ-condensing* on a subset K of E if it is ψ-condensing as a mapping $f\colon B[x_0; \delta] \to E$ for every $x_0 \in K$ satisfying $B[f(x_0), \delta] \cap K \neq \varnothing$, where $B[x; \delta]$ is the closed ball in E of radius δ centred at x. It will be called *δ-locally χ-ψ-contracting* on K if it is χ-ψ-contracting as a mapping $f\colon B[x_0; \delta] \to E$ for every $x_0 \in K$ satisfying $B[f(x_0), \delta] \cap K \neq \varnothing$,

THEOREM 1. *Let the mapping $f\colon E \to E$ be semi-hyperbolic on $K \subset E$ with a split $\mathbf{s} = (\lambda_s, \lambda_u, \mu_s, \mu_u)$ and constants k and δ. Then f is δ-locally, λ_s-ψ-contracting on K.*

The proof is divided into two steps which are presented as Lemmas 1 and 2.

Let E_1 and E_2 be Banach spaces, P_1 and P_2 be finite-dimensional projectors in E_1, E_2, and let S be a bounded subset of E_1. Finally, let $0 < \lambda < 1$. We shall call a mapping $f\colon S \to E_2$ *(P_1, P_2, λ)-contracting* if there exists a constant $a > 0$ such that for each $\varepsilon > 0$ the inequality

$$\|P_1(x-y)\|_{E_1} \leqslant \varepsilon, \qquad x, y \in S, \tag{3.1}$$

implies

$$\|(I-P_2)(f(x)-f(y))\|_{E_2} \leqslant \lambda\, \|x-y\|_{E_1} + \varepsilon a. \tag{3.2}$$

LEMMA 1. *If f is (P_1, P_2, λ)-contracting, then it is λ-ψ-contracting.*

Proof. Let V be a bounded subset of S with $\psi(V) > 0$. For a sufficiently small ε consider the sets

$$V_{j,\varepsilon} = \{x \in V\colon \|P_1(x) - a_j\| < \varepsilon\}, \tag{3.3}$$

where $\{a_j : j=1, ..., J(\varepsilon)\}$ is a finite ε-net in the finite dimensional bounded set $P_1 V$. By the definition $V = \bigcup_{j=1}^{J(\varepsilon)} V_{j,\varepsilon}$. Since ε is arbitrarily small, it is sufficient to establish the inequality

$$\psi(f(V_{j,\varepsilon})) \leqslant \lambda \psi(V_{j,\varepsilon}) + 2\varepsilon a. \tag{3.4}$$

In fact, as P_2 is finite dimensional, it suffices to show that

$$\psi((I-P_2) f(V_{j,\varepsilon})) \leqslant \lambda \psi(V_{j,\varepsilon}) + 2\varepsilon a, \qquad j=1, ..., J(\varepsilon). \tag{3.5}$$

On the other hand, from (3.3) it follows that $\|P_1(x-y)\|_{E_1} \leqslant 2\varepsilon$ for x, $y \in V_{j,\varepsilon}$, and from the definition of (P_1, P_2, λ)-contraction

$$\|(I-P_2)(f(x)-f(y))\|_{E_2} \leqslant \lambda \|x-y\|_{E_1} + 2\varepsilon a, \qquad x, y \in V_{j,\varepsilon}. \tag{3.6}$$

Let $\{s_m\}$ be a finite ξ-net in $V_{j,\varepsilon}$. Let $z \in (I-P_2) f(V_{j,\varepsilon})$, then $z = (I-P_2) f(y)$ for some $y \in V_{j,\varepsilon}$ and there exists $s_m \in V_{j,\varepsilon}$ such that $\|y - s_m\| < \xi$. Then by (3.6) and linearity of $I-P_2$ we have

$$\|(I-P_2) f(y) - (I-P_2) f(s_m)\| = \|(I-P_2)(f(y)-f(s_m))\| \leqslant \lambda\xi + 2\varepsilon a.$$

This shows that $\{(I-P_2) f(s_m)\}$ is a finite $(\lambda\xi + 2\varepsilon a)$-net in $(I-P_2) f(V_{j,\varepsilon})$ and hence that $\psi((I-P_2) f(V_{j,\varepsilon})) \leqslant \lambda\xi + 2\varepsilon a$. By property (5) of ψ, $V_{j,\varepsilon}$ has a finite ξ-net for every $\xi = \xi_n \in (\psi(V_{j,\varepsilon}), \psi(V_{j,\varepsilon}) + 1/n)$ for each $n = 1, 2, ...,$ so

$$\psi((I-P_2) f(V_{j,\varepsilon})) \leqslant \lambda \left(\psi(V_{j,\varepsilon}) + \frac{1}{n} \right) + 2\varepsilon a.$$

Taking $n \to \infty$, this estimate implies (3.5) and the lemma is proved. ∎

LEMMA 2. *If f is* **s**-*semi-hyperbolic on K, then f is $(P^u_{x_0}, P^u_{y_0}, \lambda_s)$-contracting for every $x_0, y_0 \in K$ with $\|f(x_0) - y_0\| \leqslant \delta$.*

Proof. For a given $\varepsilon > 0$, let $\|P^u_{x_0}(x-y)\| \leqslant \varepsilon$. By the definition of semi-hyperbolicity

$$\|P^s_{y_0}(f(x)-f(y))\| \leqslant \lambda_s \|P^s_{x_0}(x-y)\| + \mu_s \|P^u_{x_0}(x-y)\|,$$

so using the fact that $\|P^s_{x_0}(x-y)\| \leqslant \|x-y\| + \|P^u_{x_0}(x-y)\|$, we obtain

$$\begin{aligned} \|(I-P^u_{y_0})(f(x)-f(y))\| &\leqslant \lambda_s \|x-y\| + (\lambda_s + \mu_s) \|P^u_{x_0}(x-y)\| \\ &= \lambda_s \|x-y\| + (\lambda_s + \mu_s)\varepsilon, \end{aligned}$$

which proves Lemma 2. ∎

Remark. A **s**-semi-hyperbolic mapping f on an invariant set K, which is always locally λ_s-ψ-contracting on K by virtue of Theorem 1, need not be ψ-condensing on K. An example is given in [2].

4. LINEAR SEMI-HYPERBOLIC OPERATORS

A linear mapping $A: E \to E$ is in fact χ-ψ-contracting on E if it is δ-locally χ-ψ-contracting on some ball $B[x_0; \delta]$, $\delta > 0$, in E in view of the homogeneity property of ψ. Hence, Theorem 1 and explicit estimates on the spectrum of A give the following classification of a linear semi-hyperbolic mapping. In what follows

$$A(r_1, r_2) = \{z \in \mathbb{C}: r_1 \leqslant |z| \leqslant r_2\}, \qquad A_o(r_1, r_2) = \{z \in \mathbb{C}: r_1 < |z| < r_2\}$$

will denote the closed annulus and the open annulus in the complex plane centred on the origin with interior and exterior radii r_1 and r_2 respectively, and $r(A)$ and $\sigma(A)$ will denote the spectral radius and the spectrum of the operator A respectively.

THEOREM 2. *Let $A: E \to E$ be a continuous linear operator. If*

1. *A is χ-ψ-contracting and has no eigenvalues in the annulus $A(w_-, w_+)$, where $\chi \leqslant w_- < 1 < w_+$, then A is semi-hyperbolic on all of E with a split $(w_-, w_+, 0, 0)$;*

2. *A is **s**-semi-hyperbolic on the singleton set $\{0\}$ with a split $\mathbf{s} = (\lambda_s, \lambda_u, \mu_s, \mu_u)$, then A is λ_s-ψ-contracting and has no spectral value in the open annulus $A_o(w_-, w_+)$, where $w_\pm$ are defined by*

$$w_\pm = 1 \pm \min\{1, \tfrac{1}{2}(\lambda_u - \lambda_s - \sqrt{(\lambda_u - \lambda_s)^2 - 4(1-\lambda_s)(\lambda_u - 1) + 4\mu_s\mu_u})\}. \tag{4.1}$$

Remark. Part 1 of Theorem 2 above assumes that the operator A has no eigenvalues in the annulus $A(w_-, w_+)$, but any other kind of spectral value is, in principle, possible.

Proof of Theorem 2, *Part* 1. The proof requires the following lemmas. The first is a classical assertion (see Proposition 9.6, [5], p. 83), while the second is essentially known but for the reader's convenience we will show in the Appendix how it follows from Theorem 9.11 in [5].

LEMMA 3. *Let A be a continuous linear operator on a Banach space $(E, \|\cdot\|_E)$. Then for every $\varepsilon > 0$ there exists an equivalent norm $\|\cdot\|_\varepsilon$ on E such that $\|A\|_\varepsilon \leqslant r(A) + \varepsilon$.*

LEMMA 4. *Let $A: E \to E$ be a continuous linear operator which is χ-ψ-contracting on E with $0 \leqslant \chi < 1$. Then each $\lambda \in \sigma(A)$ with $|\lambda| > \chi$ is an eigenvalue of A.*

Let A be χ-ψ-contracting without any eigenvalues in the annulus $A(w_-, w_+)$, where $\chi \leqslant w_- < 1 < w_+$. Since A is continuous the spectrum $\sigma(A)$ of A is a compact set and consists of two disjoint sets $\sigma(A) = \sigma^s(A) \cup \sigma^u(A)$, one in each of the two parts of the exterior of the annulus. By Lemma 4 the continuous spectrum of A lies in the disk $|z| \leqslant \chi$ in the complex plane $\mathbb{C}$ and since, by assumption, no eigenvalues lie in the annulus $A(w_-, w_+)$, the component $\sigma^u(A)$ consists entirely of eigenvalues and lies in exterior region $|z| > w_+$ of the complex plane, while $\sigma^s(A)$ is located strictly inside the disk $|z| < w_-$ of the complex plane. By the Riesz–Nagy Decomposition Theorem ([21], page 421), the space E can be decomposed into the direct sum

$$E = E^s \oplus E^u \tag{4.2}$$

for which the linear subspaces E^s and E^u are invariant under A and the relationships

$$\sigma(A|_{E^s}) = \sigma^s(A), \qquad \sigma(A|_{E^u}) = \sigma^u(A) \tag{4.3}$$

hold. In particular, the unstable subspace E^u given by (4.2) is finite dimensional here, for if it were not then the restriction $A|_{E^u}$ would be an expanding (in an appropriate norm) infinite dimensional operator in contradiction to the condensing property. The Decomposition Theorem also asserts the existence of bounded projection operators P^s of E onto E^s and P^u of E onto E^u.

In view of its compactness, $\sigma(A)$ is strictly separated from the annulus $A(w_-, w_+)$, so Lemma 3 can be applied to provide the existence of norms $\|\cdot\|_{E^s}$ and $\|\cdot\|_{E^u}$ on the subspaces E^s and E^u, respectively, which are equivalent to the original norm and for which the inequalities

$$\|A|_{E^s}\|_{E^s} \leqslant w_-, \qquad \|A|_{E^u}\|_{E^u} \geqslant w_+ \tag{4.4}$$

are satisfied. Introduce an equivalent norm on E defined by

$$\|x\| = \max\{\|P^s x\|_{E^s}, \|P^u x\|_{E^u}\}. \tag{4.5}$$

The operator A is then semi-hyperbolic on E with this norm, the split $(w_-, w_+, 0, 0)$, and the constant splitting $E = E^s_x \oplus E^u_x \equiv E^s \oplus E^u$, $x \in E$, with the corresponding projectors. This completes the proof of Part 1 of Theorem 2. ∎

Proof of Theorem 2, *Part* 2. The proof of Theorem 2, Part 2 is straightforward, but somewhat lengthy and will thus be given in the appendix.

An important corollary of Theorem 2 concerns the robustness of semi-hyperbolic linear mappings to small nonlinear perturbations.

COROLLARY 1. *Let $A: E \to E$ be a continuous linear operator which is ψ-contracting and has no eigenvalues on the unit circle in $\mathbb{C}$. Then the mapping $A_\varepsilon = A + \varepsilon h$, where h is a global Lipschitz mapping, is semi-hyperbolic and ψ-contracting for all sufficiently small $\varepsilon > 0$.*

5. ψ-BISHADOWING

A mapping $f: E \to E$ on a Banach space $(E, \|\cdot\|_E)$ can be identified with a discrete-time dynamical system on the state space E generated by f through iteration,

$$x_{n+1} = f(x_n). \tag{5.1}$$

A sequence (or contiguous sequence segment) $\mathbf{x} = \{x_n\}_{n=-N_-}^{N_+} \subset E$ satisfying (5.1) for $n = -N_-, ..., 0, 1, 2, ..., N_+ - 1$, where $0 \leqslant N_-, N_+ \leqslant \infty$, is called a trajectory of the dynamical system f, while a sequence $\mathbf{y} = \{y_n\}_{n=-N_-}^{N_+} \subset E$ with

$$\|y_{n+1} - f(y_n)\|_E \leqslant \gamma, \qquad \gamma > 0, \tag{5.2}$$

for such n is called a γ-*pseudo-trajectory* of the system; they will be called *finite* trajectories and pseudo-trajectories when both $N_\pm < \infty$. Let $\mathbf{Tr}(f, K, \gamma)$ denote the totality (for all possible $N_\pm$) of γ-pseudo-trajectories (5.2) which belong entirely to a subset $K \subseteq E$ and let $\mathbf{Tr}(f, K)$ denote the set of all possible trajectories of f which belong entirely to K. A true trajectory is a γ-pseudo-trajectory for any $\gamma > 0$ and can be conveniently considered to be a 0-pseudo-trajectory, so we can write $\mathbf{Tr}(f, K) = \mathbf{Tr}(f, K, 0)$.

The gist of a Shadowing Lemma [22] for a hyperbolic mapping f is that for every $\varepsilon > 0$ there exists a $\gamma > 0$ such that each γ-pseudo-trajectory $\mathbf{y}$ of f is ε-shadowed by a true trajectory $\mathbf{x}$ of f, that is for which

$$\|x_n - y_n\|_E \leqslant \varepsilon \tag{5.3}$$

for all n belonging to some contiguous set (which is usually of finite length depending on the trajectories and the parameters). This is often used to justify the validity of numerical computations of hyperbolic systems. The inverse question of whether every true trajectory can be approximated by some pseudo-trajectory is also of practical importance and motivated the introduction of the concept of *bishadowing* of finite dimensional semi-hyperbolic mappings in [7, 9]. Bishadowing involves a class of comparison mappings whose true trajectories are pseudo-trajectories of the original system. Continuous mappings were considered in [7, 9] and compact perturbation mappings in the Banach space generalization in [3]. Wider applicability is possible, in particular to neutral delay equations, if locally ψ-condensing mappings are considered.

DEFINITION 2 (ψ-bishadowing). A *mapping* $f: E \to E$ is said to be ψ-bishadowing with positive parameters α, β and δ on a subset K of E if for any given finite pseudo-trajectory $\mathbf{y} = \{y_n\} \in \mathbf{Tr}(f, K, \gamma)$ with $0 \leqslant \gamma \leqslant \beta$ and any δ-locally ψ-condensing mapping $\varphi: E \to E$ satisfying

$$\gamma + \sup_{x \in E} \|\varphi(x) - f(x)\|_E \leqslant \beta$$

there exists a trajectory $\mathbf{x} = \{x_n\} \in \mathbf{Tr}(\varphi, E)$ such that

$$\|x_n - y_n\|_E \leqslant \alpha(\gamma + \sup_{x \in E} \|\varphi(x) - f(x)\|_E) \tag{5.5}$$

for all n for which $\mathbf{y}$ is defined.

Note that if f is itself δ-locally ψ-condensing then bishadowing includes both direct and inverse shadowing by an appropriate choice of mappings and γ, including $\varphi = f$ and $\gamma = 0$.

The bishadowing theorem in [3] for a semi-hyperbolic system in a Banach space can be extended to ψ-bishadowing where the δ in the δ-locally ψ-condensing comparison mappings is the δ in the semi-hyperbolicity definition.

THEOREM 3. *Let $f: E \to E$ be a locally Lipschitz mapping which is semi-hyperbolic on a subset K of E with a split $\mathbf{s} = (\lambda_s, \lambda_u, \mu_s, \mu_u)$ and positive constants k and δ with respect to the norm $\|\cdot\|_E$ on E. Then it is ψ-bishadowing on K for δ-locally ψ-condensing comparison mappings with bishadowing parameters*

$$\alpha(\mathbf{s}, k) = k \frac{\lambda_u - \lambda_s + \mu_s + \mu_u}{(1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u} \tag{5.6}$$

and

$$\beta(\mathbf{s}, k, \delta) = \delta k^{-1} \frac{(1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u}{\max\{\lambda_u - 1 + \mu_s, 1 - \lambda_s + \mu_u\}}. \tag{5.7}$$

Sketch of Proof of Theorem 3. The proof is structurally similar to the proof in [3] where the comparison mappings were compact perturbations of the semi-hyperbolic mapping f, so only the key features and differences will be indicated here.

Suppose we have a fixed finite γ pseudo-trajectory $\mathbf{y} = \{y_n\}_{n=0}^{N}$ with

$$0 \leqslant \|f(y_n) - y_{n-1}\|_E \leqslant \gamma \leqslant \beta(\mathbf{s}, h, \delta), \qquad y_n \in K, \qquad n = 1, ...N, \quad (5.8)$$

and a δ-locally ψ-condensing system φ satisfying

$$\gamma + \sup_{x \in E} \|\varphi(x) - f(x)\|_E \leqslant \beta(\mathbf{s}, h, \delta). \quad (5.9)$$

We need to construct a proper trajectory $\mathbf{x}$ of φ satisfying

$$\|x_n - y_n\|_E \leqslant \alpha(\mathbf{s}, h)(\gamma + \sup_{x \in E} \|\varphi(x) - f(x)\|_E), \qquad n = 0, 1, ..., N. \quad (5.10)$$

Denote by $B^u_x[0, r]$ the closed ball of radius r centred at 0 in the linear space E^u_x and for each $x, y \in K$ with $\|f(x) - y\|_E \leqslant \delta$ and for each $z \in E$ satisfying $\|P^s_x z\|_E \leqslant \delta$ define the mapping $F_{x, y, z}: B^u_x[0, \delta] \to E^u_y$ by

$$F_{x, y, z}(v) = P^u_y(f(x + P^s_x z + v) - f(x + P^s_x z)). \quad (5.11)$$

By the principle of domain invariance (see, e.g. [1], p. 396) and because the finite dimensional subspaces E^u_x, $x \in K$, have the same dimension by the property SH0, the following assertion is valid.

LEMMA 5. *Let $x, y \in K$ with $\|f(x) - y\|_E \leqslant \delta$ and $z \in E$ satisfy $\|P^s_x z\|_E \leqslant \delta$. Then $F_{x, y, z}(B^u_x[0, r]) \supseteq B^u_y[0, \lambda_u r]$ for $0 \leqslant r \leqslant \delta$.*

From this lemma and from inequality (2.6) we immediately obtain

COROLLARY 2. *Under conditions of Lemma* 5 *the operator $Q_{x, y, z} = F^{-1}_{x, y, z}$ is defined and continuous on $B^u_y[0, \lambda_u \delta]$ and satisfies $\|Q_{x, y, z}(v)\|_E \leqslant \lambda_u^{-1} \|v\|_E$.*

Consider the Banach space $\mathscr{Z}_{N+1} \cong E^{N+1}$ of $(N+1)$-tuples $\mathbf{z} = (z_0, z_1, ..., z_N)$ with components $z_j \in E$ for $j = 0, 1, ..., N$ and with the norm $\|\mathbf{z}\|_{E^{N+1}} = \max_{0 \leqslant n \leqslant N} \|z_n\|_E$. Introduce the operator $H: \mathscr{Z}_{N+1} \to \mathscr{Z}_{N+1}$ which transforms a given $\mathbf{z} \in \mathscr{Z}_{N+1}$ into $H(\mathbf{z}) = \mathbf{w} = (w_0, w_1, ..., w_N) \in \mathscr{Z}_{N+1}$ defined by the boundary conditions

$$P^s_{x_0} w_0 = 0, \qquad P^u_{x_N} w_N = 0 \quad (5.12)$$

and the relations

$$P^s_{y_n} w_n = P^s_{y_n}(\varphi(y_{n-1} + z_{n-1}) - y_n), \tag{5.13}$$

$$\begin{aligned} P^u_{y_{n-1}} w_{n-1} = Q_{y_{n-1}, y_n, z_{n-1}}(P^u_{y_n}(-\varphi(y_{n-1} + z_{n-1}) + f(y_{n-1} + z_{n-1}) \\ + y_n - f(y_{n-1} + P^s_{y_{n-1}} z_{n-1}) + z_n)) \end{aligned} \tag{5.14}$$

for $n = 1, ..., N-1, N$. Introduce the set

$$S(\beta) = \{\mathbf{z} \in \mathscr{X}_{N+1} \colon \|P^s_{y_n} z_n\|_E \leqslant a\beta \text{ and } \|P^u_{y_n} z_n\|_E \leqslant b\beta, n = 0, 1, ..., N\}, \tag{5.15}$$

where a and b are defined by

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 - \lambda_s & \mu_s \\ \mu_u/\lambda_u & 1 - 1/\lambda_u \end{pmatrix}^{-1} \begin{pmatrix} k \\ k/\lambda_u \end{pmatrix} \tag{5.16}$$

and $\beta = \gamma + \sup_{x \in E} \|\varphi(x) - f(x)\|_E$. By the definition and the split inequalities (2.1), (2.2) a and b are positive,

$$a\beta \leqslant a\beta(\mathbf{s}, k, \delta) \leqslant \delta, \qquad b\beta \leqslant b\beta(\mathbf{s}, k, \delta) \leqslant \delta, \tag{5.17}$$

and

$$\max_{\mathbf{z} \in S(\beta)} \|z_n\|_E \leqslant \alpha(\mathbf{s}, k)(\gamma + \sup_{x \in E} \|\varphi(x) - f(x)\|_E) \tag{5.18}$$

for $n = 0, 1, ..., N$.

As in [3], from the above inequalities we can show that H maps the set $S(\beta)$ continuously into itself and that a sequence $\mathbf{x} = \{y_0 + z_0, y_1 + z_1, ..., x_N + z_N\}$ is a trajectory of φ whenever $\mathbf{z} = (z_0, z_1, ..., z_N)$ is a fixed point of H in $S(\beta)$. Such a trajectory satisfies (5.10) by virtue of (5.18).

We need to prove that the mapping H is ψ-condensing on $S(\beta)$. For this it suffices to establish the inequality

$$\psi(H(M)) < \psi(M) \tag{5.19}$$

for any subset $M \subseteq S(\beta)$ with $\psi(M) > 0$. For each $n = 0, 1, ..., N$ write $M_n = \{z_n \colon \mathbf{z} \in M\}$ and $H(M)_n = \{z_n \colon \mathbf{z} \in H(M)\}$. Note that

$$\psi(M) = \max_{0 \leqslant n \leqslant N} \psi(M_n), \qquad \psi(H(M)) = \max_{0 \leqslant n \leqslant N} \psi(H(M)_n). \tag{5.20}$$

So to establish the inequality (5.19) we need to establish

$$\max_{0 \leqslant n \leqslant N} \psi(H(M_n)) < \max_{0 \leqslant n \leqslant N} \psi(M_n). \tag{5.21}$$

For each $n = 0, 1, ..., N$ let $M_n^s = \{P_{y_n}^s(z_n)\colon \mathbf{z} \in M\}$ and $H(M)_n^s = \{P_{y_n}^s(z_n)\colon \mathbf{z} \in H(M)\}$. By the definition of the Hausdorff measure of noncompactness and finite-dimensionality of the $E_{y_n}^u$, we have $\psi(M_n) = \psi(M_n^s)$ and $\psi(H(M)_n) = \psi(H(M)_n^s)$. So inequality (5.21) reduces to

$$\max_{0 \leqslant n \leqslant N} \psi(H(M)_n^s) < \max_{0 \leqslant n \leqslant N} \psi(M_n^s). \tag{5.22}$$

From the inequality (5.17), the inequalities $\|f(y_{n-1}) - y_n\|_E \leqslant \delta$ and the fact that φ is δ-locally ψ-condensing the inequalities

$$\psi(H(M)_n^s) < \psi(M_{n-1}^s) \tag{5.23}$$

hold for every $n = 1, 2, ..., N$ for which $\psi(M_n^s) > 0$. On the other hand, the first of the boundary conditions (5.12) implies that

$$\psi(H(M)_0^s) = 0. \tag{5.24}$$

Formulas (5.23) and (5.24) imply (5.22) and so the inequality (5.19) is established. Hence by a fixed point theorem for condensing mappings ([5], Theorem 9.1) H has a fixed point in $S(\beta)$. ∎

A useful restriction of ψ–bishadowing involves χ-ψ-contracting perturbations of f.

Definition 3 ((α, β, χ)-bishadowing). A mapping $f\colon E \to E$ is said to be (α, β, χ)-*bishadowing* on a subset K of E if for any finite pseudo-trajectory $\mathbf{y} = \{y_n\} \in \mathbf{Tr}(f, K, \gamma)$ with $0 \leqslant \gamma \leqslant \beta$ and any χ-ψ-contracting mapping $h\colon E \to E$ satisfying

$$\gamma + \sup_{x \in E} \|h(x)\|_E \leqslant \beta$$

there exists a trajectory $\mathbf{x} = \{x_n\} \in \mathbf{Tr}(f + h, E)$ such that

$$\|x_n - y_n\|_E \leqslant \alpha(\gamma + \sup_{x \in E} \|h(x)\|_E)$$

for all n for which $\mathbf{y}$ is defined.

Theorem 3 and Corollary 1 combine to imply:

Corollary 3. *Let $f\colon E \to E$ be* **s**-*semi-hyperbolic on a subset K of E with constants k and δ. Then it is* (α, β, χ)-*bishadowing on K with α and β given by* (5.6) *and* (5.7) *and for any $\chi < 1 - \lambda_s$.*

This corollary includes direct and inverse shadowing with respect to uniformly small perturbations with "reasonable" Lipschitz constants, as well as to uniformly small completely continuous perturbations.

Note that Theorem 3 and Corollary 3 are interesting even for a linear mapping f, in which case they can be considered as a modification of the Hartman–Grobman theorem concerning the structural proximity of a linear system and its perturbations. The conclusion is weaker in that we assert only a proximity of the sets of trajectories and not the existence of a homeomorphism between them, but the class of perturbations is much wider here.

6. APPLICATION TO NEUTRAL DELAY EQUATIONS

The preceding results will now be applied to the shift operators of neutral delay equations. In particular, the linear neutral delay equation

$$x'(t) = Cx'(t-h) + Ax(t-h) + Bx(t), \qquad t \in \mathbb{R}, \tag{6.1}$$

where $x(t)$ is an element of $\mathbb{R}^d$ with a fixed norm $|\cdot|$, h is a positive constant and A, B, C are $d \times d$-matrices will be considered along with nonlinear perturbations of (6.1) of the form

$$y'(t) = Cy'(t-h) + Ay(t-h) + By(t) + F(y(t), y(t-h), y'(t-h)), \tag{6.2}$$

where F: $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous, uniformly bounded and locally Lipschitz in its first variable.

Let $\mathscr{C}^1$ be the Banach space of all continuously differentiable functions $\phi : [-h, 0] : \to \mathbb{R}^d$ endowed with the norm

$$\|\phi\|_{\mathscr{C}^1} = \|\phi\|_{\mathscr{C}} + \|\phi'\|_{\mathscr{C}},$$

where $\|\phi\|_{\mathscr{C}} = \max_{t \in [-h, 0]} |\phi(t)|$ (the derivatives at the endpoints $t = -h$ and $t = 0$ are the left and right sided derivatives, respectively). Following Hale [1], for each $\phi \in \mathscr{C}^1$ the nonlinear equation equation (6.2) has a unique solution $y(t; \phi, F)$ for $t \geqslant -h$, which is continuously differentiable everywhere except possibly at the points $t = nh$ for $n = -1, 0, 1, 2, ...,$ and satisfies (6.2) for all $t > 0$ and the initial condition $y(t; \phi, F) = \phi(t)$ for $t \in [-h, 0]$. Let $L(F)$ denote the set of all such solutions and $L(0)$ the corresponding set of solutions $x(t; \phi)$ of the linear equation (6.1).

Recall that the linear neutral equation (6.1) is said to be *hyperbolic* if its characteristic equation

$$\det g(\omega) = 0, \tag{6.3}$$

where $g(\omega) = C\omega e^{-\omega h} + Ae^{-\omega h} + B - \omega I$, has no roots on the imaginary axis of the complex plane $\mathbb{C}$.

THEOREM 4. *Suppose that the linear Equation* (6.1) *is hyperbolic and that F is continuous, uniformly bounded, locally Lipschitz in its first variable and uniformly Lipschitz in its third variable with Lipschitz constant $\gamma_1 < 1 - |C|$, where $|C| = \max_{|x|=1} |Cx| < 1$. Then there is a constant $\gamma_2 > 0$ such that*:

(a) *For each $x(\cdot) \in L(0)$ and each positive integer N there exists a solution $y(\cdot) \in L(F)$ satisfying*

$$|y(t) - x(t)| + |y'(t) - x'(t)| < \gamma_2 \sup_{u, v, w \in \mathbb{R}^d} |F(u, v, w)| \tag{6.4}$$

for $t \in [-h, Nh]$, $t \neq 0, h, ..., (N-1)h$.

(b) *For each solution $y(\cdot) \in L(F)$ there exists a solution $x(\cdot) \in L(0)$ satisfying* (6.4) *for $t \in [-h, Nh]$, $t \neq 0, h, ..., (N-1)h$.*

The proof follows from Theorem 3 on ψ-bishadowing applied to the shift operators of the linear and nonlinear delay Equations (6.1) and (6.2). The shift operator Sh: $\mathscr{C}^1 \to \mathscr{C}^1$ for (6.1) is defined by $(\mathrm{Sh}\,\phi)(t) = x(t+h; \phi)$ for $-h \leqslant t \leqslant 0$, and its counterpart $(\mathrm{Sh}_F \phi)$ for (6.2) is defined analogously.

LEMMA 6. *The shift operator* Sh *is a linear, bounded $|C|$-ψ-contracting operator on $\mathscr{C}^1$ and the shift operator* Sh_F *is a continuous $(|C| + \gamma_1)$-ψ-contracting operator on $\mathscr{C}^1$.*

Proof. The linearity of Sh follows from that of the Equation (6.1). A constant $\mathscr{K}$ can be found with the property that

$$\|\mathrm{Sh}\,\phi\|_{\mathscr{C}^1} \leqslant \mathscr{K}\ \|\phi\|_{\mathscr{C}^1}, \tag{6.5}$$

see Chapter 7 of [11]. Hence Sh is a bounded operator on $\mathscr{C}^1$.

It remains to prove that Sh is $|C|$-ψ-contracting. Choose an arbitrary $\phi \in C^1$ and denote $\bar{x}(t) = \mathrm{Sh}\,\phi(t) = x(t+h; \phi)$, $-h \leqslant t \leqslant 0$. Integrating (6.1) gives

$$\bar{x}(t) = \phi(0) + C\phi(t) + \int_{-h}^{t} A\phi(s)\, ds + \int_{-h}^{t} B\bar{x}(s)\, ds \tag{6.6}$$

for $-h \leqslant t \leqslant 0$. Let $\mathscr{M}$ be a bounded subset of $\mathscr{C}^1$ with bound $\sup_{\phi \in \mathscr{M}} \|\phi\|_{\mathscr{C}^1} = M < \infty$ and define the subsets

$$\mathscr{B}_0 = \left\{ \bar{z} \in \mathscr{C}^1 : \bar{z}(t) = \phi(0) + A \int_{-h}^{t} \phi(s)\, ds, \phi \in \mathscr{M} \right\},$$

$$\mathscr{B}_{\mathscr{K}} = \{ \bar{x} \in \mathscr{C}^1 : \|\bar{x}\|_{\mathscr{C}^1} \leqslant \mathscr{K} M \}$$

and

$$\mathscr{B}_1 = \left\{ \bar{y} \in \mathscr{C}^1 : \bar{y}(t) = B \int_{-h}^{t} \bar{x}(s)\, ds,\ \bar{x} \in \mathscr{B}_{\mathscr{K}} \right\}.$$

Note that $\mathscr{B}_0$ and $\mathscr{B}_1$ are relatively compact in $\mathscr{C}^1$. Let $\bar{x} \in \mathrm{Sh}\,\mathscr{M}$, so $\bar{x} = \mathrm{Sh}\,\phi$ for some $\phi \in \mathscr{M}$ and by the bound (6.5), $\bar{x} \in \mathscr{B}_{\mathscr{K}}$. Now $\bar{x}$ satisfies Equation (6.6), so $\mathrm{Sh}\,\mathscr{M} \subseteq C\mathscr{M} + \mathscr{B}_0 + \mathscr{B}_1$. Since the measure of non-compactness ψ is subadditive and homogeneous, it follows then that $\psi(\mathrm{Sh}\,\mathscr{M}) \leqslant |C|\,\psi(\mathscr{M})$. Hence the shift operator Sh is $|C|$-ψ-contracting. The "nonlinear" part can be proved in a similar way. ▮

LEMMA 7. *If Equation* (6.1) *is hyperbolic, then the shift operator Sh is semi-hyperbolic in* $\mathscr{C}^1$.

Proof. An eigenfunction ϕ of the complexification of the shift operator Sh with a complex eigenvalue ω satisfies $\omega\phi'(t) = C\phi'(t) + A\phi(t) + B\omega\phi(t)$ for $-h \leqslant t \leqslant 0$, so the set of nonzero eigenvalues of the linear operator Sh coincides with the set of complex numbers $z = e^{h\omega}$ where ω is a solution of the characteristic equation (6.3). The proof is completed by applying Lemma 6 and Theorem 2. ▮

It is easy to show that the operators Sh and Sh_F satisfy

$$\|\mathrm{Sh}_F\phi - \mathrm{Sh}\,\phi\|_{\mathscr{C}^1} < \gamma_3 \sup_{u,\, v,\, w \in \mathbb{R}^d} |F(u, v, w)|,$$

for some positive γ_3 and all $\phi \in \mathscr{C}^1$.

By the "nonlinear" part of Lemma 6 and Lemma 7, Theorem 3 applied to $f = \mathrm{Sh}$ and $\varphi = \mathrm{Sh}_F$ then gives

COROLLARY 4. *There exists a constant* $\gamma_4 > 0$ *such that*:

(a) *For each trajectory* $\eta = \{\eta_0, \eta_1, ...\}$ *of the shift operator* Sh *and for each positive integer N there exists a trajectory* $\eta^F = \{\eta_0^F, \eta_1^F, ...\}$ *of the nonlinear shift opertor* Sh_F *in* $\mathscr{C}^1$ *satisfying*

$$\|\eta_n - \eta_n^F\|_{\mathscr{C}^1} \leqslant \gamma_4 \sup_{u,\, v,\, w \in \mathbb{R}^d} |F(u, v, w)|, \qquad n = 0, 1, ..., N. \tag{6.7}$$

(b) *For each trajectory* η^F *of the nonlinear shift operator* Sh_F *and for each positive integer N there exists a trajectory* η *of the linear shift operator* Sh *for which* (6.7) *is satisfied.*

Note that the iterates of the linear shift operator satisfy $(\mathrm{Sh}^n\,\phi)(t) = x(t + nh, \phi)$ for $-h \leqslant t \leqslant 0$ and $n = 0, 1, ...$ Hence Theorem 4 follows from the above corollary.

Remark. We have considered one of the simplest applications of Theorems 2 and 3. They can be applied in much the same way to other kinds of problems with condensing mappings such as differential equations of the form $x'(t) = kx'(mt)$ [4, 16], systems with infinite delays [12] and control systems with control with respect to derivatives [18], etc. The results are also applicable to the analysis of robustness of chaotic behaviour in infinite dimensional systems with homoclinic structures, in particular in delay systems [14].

7. APPENDIX

Proof of Lemma 4. For each $\varepsilon > 0$ with $\chi + \varepsilon < 1$ and for each bounded subset M of E, we have

$$\psi[(\chi+\varepsilon)^{-1} A(M)] = (\chi+\varepsilon)^{-1} \psi[A(M)] \leqslant (\chi+\varepsilon)^{-1} \chi\psi(M) < \psi(M),$$

which implies that the operator $(\chi+\varepsilon)^{-1} A$ is ψ-condensing. Hence Theorem 9.11 in [5] implies the existence of a continuous linear operators T_1 and T_2 such that $(\chi+\varepsilon)^{-1} A = T_1 + T_2$ with the T_1 finite-dimensional and $r(T_2) < 1$, where $r(T_2)$ is the spectral radius of T_2. Then we can write $A = A_1 + A_2$ where $A_1 = (\chi+\varepsilon) T_1$ is finite–dimensional and $A_2 = (\chi+\varepsilon) T_2$ has the property that

$$r(A_2) = r((\chi+\varepsilon) T_2) = (\chi+\varepsilon) r(T_2) < 1. \tag{7.1}$$

Now let $\lambda \in \sigma(A_1 + A_2)$ be such that $|\lambda| > \chi$. Then the operator $B = I - (\lambda I - A_2)^{-1} A_1$ (which is correctly defined by (7.1)) is not invertible, for otherwise the operator

$$B^{-1}(\lambda I - A_2)^{-1} = (I - (\lambda I - A_2)^{-1} A_1)^{-1} (\lambda I - A_2)^{-1}$$

would be the inverse to $\lambda I - A_1 - A_2$, which contradicts $\lambda \in \sigma(A_1 + A_2)$. Since $(\lambda I - A_2)^{-1} A_1$ is finite dimensional together with A_1, the noninvertibility of B implies that 1 is an eigenvalue of $(\lambda I - A_2)^{-1} A_1$, (see e.g. Theorem 8.9(s2) in [5], p. 63). That is, $(\lambda I - A_2)^{-1} A_1 x = x$ for some nonzero x. Therefore, λ is an eigenvalue of $A = A_1 + A_2$ with eigenvector x. ∎

Proof of Theorem 2, *Part* 2. Let A be semi-hyperbolic with a split $\mathbf{s} = (\lambda_s, \lambda_u, \mu_s, \mu_u)$ and positive constants k and δ. By Theorem 1 A is δ-locally λ_s-ψ-contracting and, since A is linear, it is thus λ_s-ψ-contracting by the homegeneity property of the measure of noncompactness ψ.

On the other hand, there is a decomposition $E = E_0^s \oplus E_0^u$ of E with projectors P^s and P^u such that $y := P^s x$, $z := P^u x$ and $x = y + z$. To finish the proof of this part of the theorem we need to establish the required estimate on the spectral values of A. To do so we must establish that any given $\lambda \in A_o(w_-, w_+)$ belongs to the resolvent set of the operator A. To this end it is sufficient to prove that for each $a \in E$ the equation

$$(\lambda I - A)x = a \tag{7.2}$$

has a unique solution $x = x(a)$ which depends continuously on a. Clearly, equation (7.2) is equivalent to the system of equations

$$\lambda y - P^s Ay - P^s Az = P^s a \qquad \lambda z - P^u Ay - P^u Az = P^u a. \tag{7.3}$$

The space E_0^u is invariant for the restriction $D := P^u A|_{E_0^u}$ of the operator $P^u A$ to E_0^u. By the definition of semi-hyperbolicity, the space E_0^u is finite dimensional, and since D is expansive with respect to the norm $\|\cdot\|_E$, D^{-1} exists as an operator from E_0^u into E_0^u. Hence the system (7.3) is equivalent to the system

$$\begin{aligned} y &= \frac{1}{\lambda} P^s Ay + \frac{1}{\lambda} P^s Az + \frac{1}{\lambda} P^s a \\ z &= -D^{-1} P^u Ay + \lambda D^{-1} z - D^{-1} P^u a. \end{aligned} \tag{7.4}$$

For $y \in E_0^s$, $z \in E_0^u$ define the linear operator $\tilde{A}$: $E_0^s \times E_0^u \to E_0^s \times E_0^u$ by

$$\tilde{A} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} S_s & S_u \\ L_s & L_u \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}, \tag{7.5}$$

where S_s (resp. S_u) is the restriction of the operator $(1/\lambda) P^s A$ to the space E_0^s (resp. E_0^u) and L_s (resp. L_u) is the restriction of the operator $-D^{-1} P^u A$ (resp. λD^{-1}) to the space E_0^s (resp. E_0^u). Define also the operator $\tilde{B}_a$: $E_0^s \times E_0^u \to E_0^s \times E_0^u$ by

$$\tilde{B}_a \begin{pmatrix} y \\ z \end{pmatrix} = \tilde{A} \begin{pmatrix} y \\ z \end{pmatrix} + \begin{pmatrix} (1/y)\, P^s a \\ -D^{-1} P^u a \end{pmatrix}. \tag{7.6}$$

System (7.4) can be now rewritten as

$$\tilde{B}_a \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} y \\ z \end{pmatrix}. \tag{7.7}$$

For a given $\gamma > 0$ define a norm $|\binom{a}{b}|_\gamma = \max\{|a|, \gamma\, |b|\}$ on $\mathbb{R}^2$ and define the auxiliary norm $\|\cdot\|_\gamma$ on $E = E_0^s \times E_0^u$ by

$$\|x\|_\gamma = \max\{\|P^s x\|, \gamma\, \|P^u x\|\} = |(\|P^s x\|, \gamma\, \|P^u x\|)^T|_\gamma.$$

For $y_1, y_2 \in E_0^s$, $z_1, z_2 \in E_0^u$, $\lambda \in A_o(w_-, w_+), 0 \leqslant \tau < 1$, from the semi-hyperbolicity and linearity of A we obtain

$$\left\|\begin{pmatrix} S_s(y_1 - y_2) + S_u(z_1 - z_2) \\ L_s(y_1 - y_2) + L_u(z_1 - z_2) \end{pmatrix}\right\|_\gamma \leqslant \left|\begin{pmatrix} \dfrac{\lambda_s}{1-\tau} & \dfrac{\mu_s}{1-\tau} \\ \dfrac{\mu_u}{\lambda_u} & \dfrac{1+\tau}{\lambda_u} \end{pmatrix}\right|_\gamma \left\|\begin{pmatrix} y_1 - y_2 \\ z_1 - z_2 \end{pmatrix}\right\|_\gamma,$$

where

$$0 \leqslant \tau < \min\{1, \tfrac{1}{2}(\lambda_u - \lambda_s - \sqrt{(\lambda_u - \lambda_s)^2 - 4(1-\lambda_s)(\lambda_u - 1) + 4\mu_s\mu_u})\}.$$

Hence

$$\left\|\tilde{B}_a \begin{pmatrix} y_1 \\ z_1 \end{pmatrix} - \tilde{B}_a \begin{pmatrix} y_2 \\ z_2 \end{pmatrix}\right\|_\gamma = \left\|\tilde{A} \begin{pmatrix} y_1 - y_2 \\ z_1 - z_2 \end{pmatrix}\right\|_\gamma \leqslant |M(\mathbf{s}, \tau)|_\gamma \left\|\begin{pmatrix} y_1 - y_2 \\ z_1 - z_2 \end{pmatrix}\right\|_\gamma, \tag{7.8}$$

where

$$M(\mathbf{s}, \tau) = \begin{pmatrix} \dfrac{\lambda_s}{1-\tau} & \dfrac{\mu_s}{1-\tau} \\ \dfrac{\mu_u}{\lambda_u} & \dfrac{1+\tau}{\lambda_u} \end{pmatrix}. \tag{7.9}$$

Since the entries of the matrix $M(\mathbf{s}, \tau)$ are positive then by the Perron–Frobenius Theorem its spectral radius

$$r(\mathbf{s}, \tau) = \frac{1}{2}\left(\left(\frac{1+\tau}{\lambda_u} + \frac{\lambda_s}{1-\tau}\right) + \sqrt{\left(\frac{1+\tau}{\lambda_u} - \frac{\lambda_s}{1-\tau}\right)^2 + \frac{4\mu_s\mu_u}{(1-\tau)\lambda_u}}\right)$$

is the maximal eigenvalue and satisfies the inequality

$$r(\mathbf{s}, \tau) < 1. \tag{7.10}$$

Moreover, the corresponding eigenvector has positive coordinates. Without loss of generality, we may assume that this eigenvector takes the form $(1, \gamma(\mathbf{s}, \tau))$ where

$$\gamma(\mathbf{s}, \tau) = \frac{1-\tau}{\mu_s} r(\mathbf{s}, \tau) - \frac{\lambda_s}{\mu_s}.$$

For a fixed $\mathbf{s}$ and τ let $\gamma = \gamma(\mathbf{s}, \tau)$, then $|M(\mathbf{s}, \tau)|_\gamma = r(\mathbf{s}, \tau)$. By (7.10) and (7.8) it follows that mapping $\tilde{B}_a$ is a contraction in the norm $\|\cdot\|_\gamma$, and by the Contraction Mapping Principle $\tilde{B}_a$ has a unique fixed point which depends continuously on a. Therefore equation (7.7) and, consequently Equation (7.2), have for each $a \in E$ a unique solution which depends continuously on a. This completes the proof Theorem 2, Part 2. ∎

Remark. A related result has been established in [15].

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